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1980 J. Phys. A: Math. Gen. 13 L235

(http://iopscience.iop.org/0305-4470/13/7/003)

J. Phys. A: Math. Gen. **13** (1980) L235–L237. Printed in Great Britain

LETTER TO THE EDITOR

A note on the two-body problem in linearised gravity

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Received 28 March 1980

Abstract. We present a novel deduction of the inverse-square law of attraction between two masses in linearised gravity, utilising the Curzon and Levi-Civita solutions of Einstein's vacuum field equations.

The Weyl static axially symmetric vacuum fields are given by (see Synge 1966) the line element

$$ds^2 = e^{2(\nu-\lambda)}(dr^2 + dz^2) + r^2 e^{-2\lambda} d\phi^2 - e^{2\lambda} dt^2 \tag{1}$$

where λ, ν are functions of r, z and Einstein's vacuum field equations reduce to the differential equation

$$d\nu = r\left[\left(\frac{\partial\lambda}{\partial r}\right)^2 - \left(\frac{\partial\lambda}{\partial z}\right)^2\right] dr + 2r\frac{\partial\lambda}{\partial r}\frac{\partial\lambda}{\partial z} dz \tag{2}$$

and its integrability condition

$$\frac{\partial^2\lambda}{\partial r^2} + \frac{1}{r}\frac{\partial\lambda}{\partial r} + \frac{\partial^2\lambda}{\partial z^2} = 0. \tag{3}$$

The Curzon (1924) solution corresponds to

$$\lambda = -m_1/p_1 - m_2/p_2, \tag{4}$$

$$\nu = -\frac{m_1^2 r^2}{2p_1^4} - \frac{m_2^2 r^2}{2p_2^4} + \frac{2m_1 m_2}{(z_1 - z_2)^2}\left(\frac{r^2 + (z - z_1)(z - z_2)}{p_1 p_2} - 1\right), \tag{5}$$

where m_1, m_2 are constant 'masses' of 'particles' located on the z axis at $z = z_1$ and $z = z_2$ respectively while $p_1^2 = r^2 + (z - z_1)^2$ and $p_2^2 = r^2 + (z - z_2)^2$. The configuration when viewed in a Euclidean space with coordinates (r, z, ϕ) is depicted in figure 1. Elementary flatness on the axis $r = 0$ requires that the limit of the ratio of the circumference c_0 to the radius r_0, as $r_0 \to 0$, of an infinitesimal space-like circle with centre on $r = 0$ be 2π. For a circle on which (r, z, t) are constant, we find for (1) that

$$c_0/r_0 = 2\pi e^{-\nu(0,z)}. \tag{6}$$

For (5) we easily see that $\nu(0, z) = 0$ for $z < z_1$ and $z > z_2$ (see figure 1), whereas

$$\nu(0, z) = -4m_1 m_2/(z_1 - z_2)^2, \qquad z_1 \leqslant z \leqslant z_2. \tag{7}$$

This is interpreted physically as the requirement of a 'strut' along AA′ to keep the

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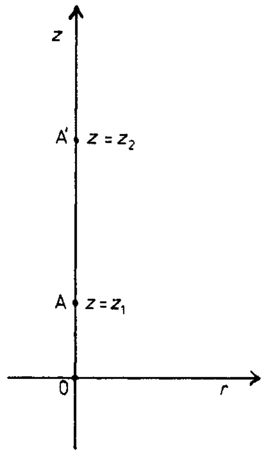

Figure 1. The two-dimensional subspace ϕ = constant of a Euclidean space with cylindrical coordinates (r, z, ϕ).

particles in position. If, using units for which $c = G = 1$, we assume that the dimensionless quantity (7) is small of first order, we may write (6) as

$$c_0/r_0 = 2\pi[1+4m_1m_2/(z_1-z_2)^2]+O_2. \qquad (8)$$

We shall regard the configuration described in figure 1 as a typical instant (frozen by the use of the 'strut' AA′) during the motion of the masses m_1, m_2 along the z axis in each others gravitational field. At such an instant the acceleration of each mass is a_1 and a_2 respectively (say). To calculate the accelerations we use the Levi-Civita (1918) solution of Einstein's vacuum field equations. This solution describes the field of a uniformly accelerating mass. In the linear approximation we will apply it to both of the masses in figure 1 at the instant when m_1 is at $z = z_1$ and m_2 is at $z = z_2$. In the form given by Robinson and Robinson (1972) the Levi-Civita solution reads

$$ds^2 = \rho^2[f^{-1}(d\xi - af\, d\sigma)^2 + f\, d\eta^2] - 2\, d\rho\, d\sigma - (K - 2H\rho - 2m\rho^{-1})\, d\sigma^2, \qquad (9a)$$

$$f = (1-\xi^2)(1+2ma\xi), \qquad (9b)$$

$$K = -\tfrac{1}{2}\, d^2f/d\xi^2, \qquad (9c)$$

$$H = -\tfrac{1}{2}a\, df/d\xi. \qquad (9d)$$

Here m is the mass of the particle and a the rectilinear acceleration (see Kinnersley and Walker 1970, Hogan and Imaeda 1979). To examine this solution for elementary flatness on its axis of symmetry (the direction of motion), we first take σ constant and in the resulting line element make the transformation (Kinnersley and Walker 1970)

$$\theta = \int_{\xi}^{+1} f^{-1/2}\, d\xi, \qquad \phi = k\eta, \qquad (10)$$

where k is a constant to be determined, and then

$$ds^2 = \rho^2(d\theta^2 + h^2(\theta)\, d\phi^2) \qquad (11)$$

where

$$h(\theta) = k^{-1}[f(\theta)]^{1/2}. \qquad (12)$$

Since we have chose σ constant we can assume that the particle is at $z = z_1$ ($\rho = 0$) in figure 1, and we will take $m = m_1$, $a = a_1$ and assume that the dimensionless quantity $m_1 a_1$ is small of first order. The ratio (6) for a small circle with centre on the z axis for $z < z_1$ ($\theta = \pi$) and radius in the $\mathrm{d}\theta$ direction can be made equal to 2π by choosing in (10)

$$k = \tfrac{1}{2}(\mathrm{d}f/\mathrm{d}\xi)_{\xi=-1}, \qquad (13)$$

whereas if the centre of the circle is a point on AA′ we find

$$c_0/r_0 = 2\pi(1+4m_1a_1) + \mathrm{O}_2. \qquad (14)$$

Comparing this with (8) we see that, *at the instant depicted in figure* 1,

$$m_1a_1 = m_1m_2/(z_1 - z_2)^2 + \mathrm{O}_2. \qquad (15)$$

Turning next to m_2, we ensure elementary flatness at points $z > z_2$ by choosing in (10)

$$k = -\tfrac{1}{2}(\mathrm{d}f/\mathrm{d}\xi)_{\xi=+1}, \qquad (16)$$

and now if the centre of the infinitesimal circle is on AA′ we have

$$c_0/r_0 = 2\pi(1-4m_2a_2) + \mathrm{O}_2, \qquad (17)$$

and comparison with (8) yields

$$m_2a_2 = -m_1m_2/(z_1 - z_2)^2 + \mathrm{O}_2 \qquad (18)$$

at the instant depicted in figure 1.

In (15) and (18) we have obtained the inverse-square law of force between two masses in the linear approximation. Cooperstock (1975) has also used the Curzon solution to study this problem, but has chosen to introduce non-static perturbations of this field which results in a considerably more complicated discussion than we have here. More recently Goldberg and Silaban (1976) have examined the problem in null coordinates attached to one of the bodies and were unable to produce the Newtonian interaction between them.

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